Robust estimates for the approximation of the dynamic consolidation problem

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Abstract. We consider stable discretizations in time and space for the linear dynamic consolidation problem describing the wave propagation in a porous solid skeleton which is fully saturated with an incompressible fluid. Introducing the hydrostatic pressure, the flow problem is described by Darcy’s law. In particular, we discuss the case of nearly impermeable solids which requires inf-sup stable discretizations in space for the limiting saddle point problem. Together with an (implicit) Newmark discretization in time we derive convergence estimates for the fully discrete scheme which are robust with respect to the coupling parameter of fluid and solid.

1. Introduction

We consider the numerical approximation of the poro-elastic system
\[ \rho \ddot{u}(t) - \nabla \cdot (\sigma(t) - p(t)I) = \rho b(t), \]  
\[ \nabla \cdot (\dot{u}(t) - \kappa \nabla p(t)) = 0 \]  
(1a)  
(1b)

describing the dynamics of a materially incompressible porous solid which is saturated by an incompressible viscous pore fluid. Here, \( u \) is the solid displacement vector, \( p \) is the hydrostatic pressure, and \( \sigma \) is the effective stress (which is related to the solid displacement by a constitutive law).

We focus our attention on the short term behavior of the system for variations in the hydraulic conductivity \( \kappa > 0 \). Of particular interest for many applications are small values of \( \kappa \). Physically, for \( \kappa = 0 \) the fluid does no longer flow through the porous solid, and the two-phase problem reduces to a single-phase problem for the displacement of the solid-fluid mixture. In this case, \( p \) admits an interpretation as the Lagrange multiplier of the constraint \( \nabla \cdot \dot{u} = 0 \). However, we do not consider the case of vanishing \( \kappa \).

In particular, we aim for robust discretizations in time and space for the full problem. Therefore, we combine two methods which are robust for its own: in time, the Newmark scheme is applied (which is unconditionally stable for a suitable choice of parameters) and in space an inf-sup stable saddle point discretization provides robust estimates. Our main result (Theorem 14) proves convergence estimates in space and time independent of \( \kappa \) and without mesh-dependent restrictions on \( \Delta t \). Numerically, this was already observed in [32] where the spatial discretization was based on Taylor-Hood elements.

The qualitative analysis of (1) is well established. Finite element approximations of the quasi-static problem (where acceleration effects are neglected) have been considered by various authors, see, e.g., [21, 24] and the references therein. The approximation of a different fully dynamic consolidation problem is analyzed by Santos et al. [29, 30]. For nonlinear models, the numerical convergence analysis in time and space is more involved already for the single phase model (see, e.g., [9] for a first order method in time applied to a dynamic model in finite elasticity). The extension of the numerical analysis to other nonlinear applications such as poro-plasticity is not done so far (see [14, 38] and the references therein for numerical simulations).

The paper is organized as follows. We start with an introduction to the modeling background in Section 2. Then we give a precise formulation of the dynamic consolidation model in Section 3, and we review the

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main properties of the problem. Then, in Section 4 the Newmark scheme is introduced, its properties and some variants are discussed. In Section 5, based on an inf-sup stable discretization in space, a robust semi-discrete estimate is derived, and finally, in Section 6 estimates for the full discretization are considered. We close with some remarks on discrete energy conservation properties of the scheme.

2. Modeling of poro-elastic phenomena

In this section we summarize modeling aspects of the system under consideration. For more details we refer to [2, 6, 11, 14, 15] and [13] for a historical overview.

Consolidation processes describe the dynamics of a system composed of a porous solid which is saturated by a viscous pore fluid. In the following, we assume that the pore space is fully saturated by the pore fluid, thus resulting in a two-phase system. We consider the case of material incompressibility of the solid and the fluid, i.e., the solid skeleton is assumed to be incompressible as well as the pore fluid. However note that despite of the material incompressibility of the individual constituents, the bulk density of the composed two-phase system (mixture) is not necessarily constant. In standard applications the solid phase consists of granular media (e.g., sand, rock, clay), the fluid phase is water. Thus, in comparison with the material behavior of the composed material, the single components indeed can be assumed to be incompressible.

A convenient way for modeling such problems is the theory of mixtures resulting in the framework of superimposed continua. The basic idea is that every spatial point of the mixture is simultaneously occupied by material points of all constituents. Moreover, to each constituent, an individual motion is assigned and as a consequence, balance equations have to be established separately for each phase \( \alpha \). For the present two-phase model, \( \alpha \in \{S, F\} \), where \( S \) denotes the solid phase and \( F \) the fluid phase. A further consequence is that the kinematics have to be established in a geometric nonlinear setting in order to affiliate the individual motions. The concept of superimposed continua gives rise to the definition of volume fractions \( n^\alpha \) in each spatial point. The volume fraction \( n^\alpha \) describes the local ratio of the volume occupied by constituent \( \alpha \) with respect to the bulk volume. Thus, we have \( n^S + n^F = 1 \).

2.1. Kinematic and balance relations. The motion assigned to phase \( \alpha \) is given by \( \chi_\alpha : \Omega \times [0, T] \rightarrow \Omega^\alpha_t \subset \mathbb{R}^d \), where the current configuration \( \Omega^\alpha_t \) is the image of the reference configuration \( \Omega \) under the action of the motion \( \chi_\alpha \) at time \( t \in [0, T] \). It is assumed that for fixed time \( t \), the mapping \( \chi_\alpha(\cdot, t) \) is bijective. We define the displacement vector \( u^\alpha(X, t) = x - X \) with \( x = \chi_\alpha(X, t) \in \Omega^\alpha_t \) for all \( X \in \Omega \) (and, by the inverse motion, \( X = \chi_\alpha^{-1}(x, t) \) for \( x \in \Omega^\alpha_t \)). This defines the deformation gradient \( F_\alpha = \text{Grad} x = \text{Grad} \chi_\alpha(X, t) \), where \text{Grad} acts on the reference configuration \( \Omega \). Likewise, the inverse of the deformation tensor is given as function of \( x \in \Omega^\alpha_t \) via \( F_\alpha^{-1} = \text{grad} X = \text{grad} \chi_\alpha^{-1}(x, t) \), where \text{grad} acts on the current configuration \( \Omega^\alpha_t \).

Then, the velocity with respect to the reference configuration is the temporal derivative of the motion \( V^\alpha(X, t) = \dot{u}^\alpha(X, t) = \frac{\partial}{\partial t} \chi_\alpha(X, t) \). With respect to the current configuration, the velocity can now be expressed by means of the inverse motion, i.e., \( \dot{v}^\alpha(x, t) = V^\alpha(\chi_\alpha^{-1}(x, t), t) \). Here, the partial derivative in time is indicated by a superposed dot. In order to describe physical processes in the deformed configuration, we also need time derivatives following the motions \( \chi_\alpha \) of the different phases. This is the material time derivative with respect to the phase \( \alpha \) and is denoted by \( (\cdot)'^\alpha \). E.g., for a scalar field \( g \), the material time derivative is given by \( (g)'^\alpha = \dot{g} + \text{grad} g \cdot \dot{v}^\alpha \). If \( g \) is a vector field, we similarly have \( (g)'^\alpha = \dot{g} + (\text{grad} g) \cdot \dot{v}^\alpha = \hat{\dot{g}} + (\psi^\alpha \cdot \text{grad}) g \).

Whereas for the solid description, the Lagrangian approach with reference configuration \( \Omega \subset \mathbb{R}^d \) is quite natural, for velocity fields an Eulerian setting is more appropriate. Here, the velocity of the fluid is evaluated relative to the current configuration (modified Eulerian setting), and our goal is to express all kinematic relations and balance laws relative to the motion of the solid phase.

The balance equation of momentum and mass for phase \( \alpha \) are given by (excluding mass exchange between the solid and fluid phase)

\[
\rho^\alpha (u^\alpha)'^\alpha - \text{div} \sigma^\alpha = \rho^\alpha b^\alpha + \hat{\rho}^\alpha, \\
(\rho^\alpha)'^\alpha + \rho^\alpha \text{div}(u^\alpha)'^\alpha = 0,
\]
where \( \rho^\alpha = n^\alpha \rho^{\alpha,0} \) is the partial density of phase \( \alpha \) and \( \rho^{\alpha,0} \) is the real material density of phase \( \alpha \) (which is constant due to the assumption of material incompressibility). Here, \( \sigma^\alpha \) is the stress tensor of phase \( \alpha \), \( b^\alpha \) is related with a volume force acting on phase \( \alpha \) and the momentum production terms \( \hat{p}^\alpha \) are constrained by \( \sum_\alpha \hat{p}^\alpha = 0 \), i.e., \( \hat{p}^S + \hat{p}^F = 0 \).

Integrating the mass balance equation of the solid phase and using the definition of the inverse deformation tensor \( F^{-1}_S \), we obtain (cf. formula (19) in [14])

\[
\rho^S = \rho^S_0 \det F^{-1}_S ,
\]

where \( \rho^S_0 = \rho^S(0) = n^S(0)\rho^{S,0} \) is the partial density of the solid in the reference configuration. The quantities \( \rho^S, \rho^F, n^S \) and \( n^F \) are completely determined by the motion of the solid skeleton as the inverse deformation tensor is given by \( F^{-1}_S = \text{grad} (x - u^S(x^S_1(x,t), t)) \). This expresses \( \rho^S \) by means of the solid displacement vector \( u^S \). Due to the material incompressibility of the solid, this determines the volume fraction of the solid phase \( n^S = \frac{\rho^S}{\rho^{S,0}} \), and by the saturation condition \( n^F = 1 - n^S \), also the volume fraction \( n^F \) is given. Finally, the material incompressibility of the fluid yields \( \rho^F = n^F \rho^{F,0} \).

The fluid balance equations can be expressed with respect to the current configuration determined by the motion of the solid phase. This allows to sum up the balance equations of the individual phases to obtain the balance equations of the mixture. Introducing the seepage velocity \( \mathbf{w} = (\mathbf{u}^F)'_F - (\mathbf{u}^S)'_S \) describing the fluid flow relative to the motion of the solid skeleton, this can be inserted in the momentum and mass balance equations to obtain

\[
\rho^F ((\mathbf{w})'_S + (\mathbf{w} \cdot \text{grad})(\mathbf{u}^S)'_S + \mathbf{w})) + (\rho^F + \rho^S)(\mathbf{u}^S)'_S - \text{div}(\sigma^S + \sigma^F) = \rho^S \mathbf{b}^S + \rho^F \mathbf{b}^F, \\
\text{div}((\mathbf{u}^S)'_S + n^F \mathbf{w}) = 0 ,
\]

(cf. equations (25) and (28) in [14]).

In the summarized mass balance equation, the temporal derivative of the partial densities vanishes due to the assumption of material incompressibility. This also implies that the mass balance equation reduces to a balance equation for the volume fractions since \( \rho^\alpha = n^\alpha \rho^{\alpha,0} \). Introducing the bulk density of the mixture \( \rho = \rho^S + \rho^F \), the stress tensor of the mixture \( \sigma^\text{mix} = \sigma^S + \sigma^F \) and assuming \( \mathbf{b}^\alpha \equiv \mathbf{b} \), we obtain the equations

\[
\rho^F ((\mathbf{w})' + (\mathbf{w} \cdot \text{grad})(\mathbf{u})' + \mathbf{w})) + \rho(\mathbf{u})'' - \text{div}\sigma^\text{mix} = \rho \mathbf{b}, \\
\text{div}(\mathbf{u}' + n^F \mathbf{w}) = 0 ,
\]

where we dropped the superscript for the solid phase, i.e., \( \mathbf{u} \equiv \mathbf{u}^S \), and the subscript for material derivatives. From now on, all material derivatives are with respect to the solid phase, i.e., \( (\cdot)' = (\cdot)'_S \). In the summarized balance equations (3), the unknowns are now the solid displacement \( \mathbf{u} \), the seepage velocity \( \mathbf{w} \) and the stress tensor of the mixture \( \sigma^\text{mix} \).

2.2. The infinitesimal case and constitutive relations. The balance equations in the full geometrically nonlinear setting simplify in the infinitesimal case. Since in most applications in soil mechanics the displacements are very small (compared with the size of the computational domain), we will now focus on the linearized equations where the nonlinear terms are neglected. In this context, material derivatives in time can be identified with the partial derivatives, i.e., \( \dot{\mathbf{u}} = (\mathbf{u})' \). Moreover, \( \text{Grad} \) and \( \text{grad} \) can be identified, and volume fractions are (approximately) constant.

Furthermore, in the following analysis, we are mainly interested in the short term behavior of the system, which allows to assume that inertial effects with respect to the relative fluid flow are small and can be neglected. This is justified when the hydraulic conductivity is of small magnitude and the fluid is forced to follow the solid motion. Thus, we consider

\[
\rho \ddot{\mathbf{u}} - \text{div}\sigma^\text{mix} = \rho \mathbf{b}, \\
\text{div}(\dot{\mathbf{u}} + n^F \mathbf{w}) = 0 ,
\]

with unknowns \( \mathbf{u}, \mathbf{w} \) and \( \sigma^\text{mix} \).
The mixture balance equations (4) have to be closed by constitutive relations. For the stress of the mixture, we use the effective stress concept of Terzaghi and impose the relation

$$\sigma^\text{mix} = \sigma - p I,$$

where $\sigma$ is the effective stress and $p$ denotes the fluid pore pressure (for a detailed analysis of the effective stress concept, also see [7]). The effective stress $\sigma$ itself can then be expressed by means of the solid deformation, and in the linear and isotropic case, the governing relation is Hooke’s law

$$\sigma = C : \varepsilon(u) = 2\mu \varepsilon(u) + \lambda \text{tr} \varepsilon(u) I = 2\mu \varepsilon(u) + \lambda \text{div}(u) I. \quad (5)$$

Here, $C$ denotes the fourth order elasticity tensor with Lamé constants $\mu, \lambda$ and the linearized Green-St. Venant strain is given by $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$.

However, an additional equation is required to close the system. This is Darcy’s filter law

$$n^F w = -\kappa \nabla p, \quad (6)$$

relating the seepage velocity to the fluid pressure. Here, $\kappa$ is the hydraulic conductivity which in general is a second order tensor. The hydraulic conductivity itself is given by $\kappa = \frac{K}{\eta}$, where $K$ is related to the permeability of the pores and $\eta$ is the fluid viscosity. Therefore, very small values of $\kappa$ can be interpreted in two ways: the pores are almost impermeable for the fluid or the large viscosity prevents fluid flow through the solid skeleton. In the limiting case $\kappa = 0$, the two-phase system reduces to a single-phase system for the displacement $u$ of the solid skeleton, and (1) reduces to the wave equation of linear elasticity with the constraint $\text{div} \dot{u} = 0$. The above constitutive relations allow to reformulate (3) by considering $(u, p)$ as the primal variables, and inserting into (4) finally yields the equations (1).

2.3. Boundary conditions. Concerning the boundary conditions on $\Gamma = \partial \Omega$, we follow the notation introduced in [34, 35]. We consider disjoint boundary decompositions $\Gamma = \Gamma_c \cup \Gamma_f$ corresponding to Dirichlet (clamped) and Neumann (traction) boundary conditions for the solid, and $\Gamma = \Gamma_d \cup \Gamma_f$, corresponding to Dirichlet (drained) and Neumann (flux) boundary conditions for the fluid. By $\Gamma_s = \Gamma_f \cap \Gamma_f$, we denote the part of the boundary, where the fluid is not drained and the solid is not clamped. On $\Gamma_s$, we can introduce a function $\chi : \Gamma_s \rightarrow [0, 1]$ determining the ratio of open pores on $\Gamma_s$. Therefore, we consider the boundary conditions

$$p = 0 \quad \text{on } \Gamma_d, \quad (7a)$$
$$\chi \dot{u} \cdot n - \kappa \nabla p \cdot n = 0 \quad \text{on } \Gamma_f, \quad (7b)$$
$$u(t) = 0 \quad \text{on } \Gamma_c, \quad (7c)$$
$$\left(\sigma - (1-\chi)p I\right) n = 0 \quad \text{on } \Gamma_t. \quad (7d)$$

We briefly discuss the two limit cases: when $\chi = 0$, then all pores are sealed and as a consequence, the sealed pores contribute to the normal stress and there is no flow across $\Gamma_s$, i.e., $\kappa \nabla p \cdot n = 0$ on $\Gamma_s$. On the other hand, if we take $\chi = 1$, all pores are open on $\Gamma_s$, the pore pressure does not contribute to the normal stress and we have non-vanishing flow across $\Gamma_s$.

2.4. Extended and related models. Historically, the investigation of poro-elasticity and particularly consolidation problems goes back to Biot’s paper on the quasi-static consolidation problem [4]. Here, based on a phenomenological understanding of porous media, he derived the fundamental equations

$$\text{div} \sigma - \nabla p = 0,$$
$$\text{div}(\dot{u} - \kappa \nabla p) = 0.$$

These equations can also be obtained in the above framework for slow processes where acceleration terms can be neglected. Thus, this model is appropriate for long term considerations. Biot’s quasi-static consolidation problem was studied extensively in the literature, see, e.g., [24, 36, 37] or the overview article [34]. Extensions of the quasi-static problem with respect to plastic deformation have been addressed numerically in [2] and [38]. Biot also proposed a dynamic consolidation problem [5] which considers the solid and fluid displacements as primary variables but which does not include mass balance equations. Biot’s dynamic consolidation problem was numerically analyzed in [29, 30].
As mentioned above, for \( \kappa = 0 \), problem (1) also admits an interpretation as the elastic wave equation for an incompressible solid and the pressure \( p \) can be interpreted as the Lagrange multiplier for the incompressibility constraint \( \text{div } \mathbf{u} = 0 \). For \( \kappa > 0 \), the system can then be interpreted as a penalty formulation.

A generalization of problem (1) is given when the saturating fluid is slightly compressible. In this case, the mass balance equations can not be reduced to a balance equation for the volume fractions, and after imposing a relation between the pressure \( p \) and the partial density of the fluid \( \rho \), the mass balance equation of the mixture now contains a time derivative of the fluid pressure. This results in the system

\[
\begin{align*}
\rho \ddot{\mathbf{u}} - \text{div} \left( \mathbf{\sigma} - \rho \mathbf{I} \right) &= \rho \mathbf{b}, \quad (8a) \\
c \dot{p} + \text{div} \left( \mathbf{\dot{u}} - \kappa \nabla p \right) &= c h, \quad (8b)
\end{align*}
\]

where \( c \) is related with the compressibility properties of the fluid and \( h \) is a source term in \( \Omega \). Such problems are investigated in \([35]\). Note that formally the above equations are equivalent to the equations modeling linear thermoelasticity (by replacing the pore pressure \( p \) by the temperature \( \theta \) and where \( c \) is the specific heat and \( \kappa \) is the thermal conductivity). In the context of thermoelasticity, well-posedness was investigated by Dafermos in \([12]\) (see also \([23, \text{Chap. 13}]\)).

3. The continuous problem

In this section we briefly recall the full equations of the consolidation problem, and we summarize the basic analytical properties. It does not contain new results; it serves for the clear definition of the problem and for the introduction of the notation.

3.1. Equations and boundary conditions. We assume that \([0, T] \subset \mathbb{R} \) is a finite time interval and \( \Omega \subset \mathbb{R}^d \) is a bounded domain with Lipschitz boundary \( \Gamma = \partial \Omega \). We assume that the Dirichlet parts have positive measure \( \text{meas}_{d-1}(\Gamma_c) \), \( \text{meas}_{d-1}(\Gamma_d) \). Depending on the boundary, we use the spaces \( \mathbf{X} = \{ \mathbf{w} \in H^1(\Omega, \mathbb{R}^d) : \mathbf{w} = 0 \text{ on } \Gamma_c \} \) for the displacements and \( Q = \{ q \in H^1(\Omega) : q = 0 \text{ on } \Gamma_d \} \) for the pressure.

Furthermore, define \( H = L_2(\Omega, \mathbb{R}^d) \) and \( H = L_2(\Omega, \mathbb{R}) \).

In \( \Omega \), the following data are given: the density of the bulk material \( \rho \in L_\infty(\Omega, \mathbb{R}) \) with \( \rho(x) \geq \rho_0 > 0 \) a.e. in \( \Omega \), the hydraulic conductivity tensor \( \kappa \in L_\infty(\Omega, \mathbb{R}^{d \times d}) \) which is uniformly positive definite, i.e., \( \xi^T \kappa(x) \xi \geq \kappa_0 |\xi|^2 \) for \( \xi \in \mathbb{R}^d \), and a volume force \( \mathbf{b} \in L_2(0, T; H) \).

Now, the full system is given by the differential equations and the boundary conditions

\[
\begin{align*}
\rho \ddot{\mathbf{u}}(t) - \text{div} \left( \mathbf{C} : \mathbf{e}(\mathbf{u}(t)) - p(t) \mathbf{I} \right) - \rho \mathbf{b}(t) &= 0 \quad \text{in } \Omega, \quad (9a) \\
\text{div} \left( \mathbf{\dot{u}}(t) - \kappa \nabla p(t) \right) &= 0 \quad \text{in } \Omega, \quad (9b) \\
p(t) &= 0 \quad \text{on } \Gamma_d, \quad (9c) \\
\kappa \nabla p(t) \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_f, \quad (9d) \\
\mathbf{u}(t) &= 0 \quad \text{on } \Gamma_c, \quad (9e) \\
(\mathbf{\sigma}(t) - p(t) \mathbf{I}) \mathbf{n} &= 0 \quad \text{on } \Gamma_t, \quad (9f)
\end{align*}
\]

subject to initial values

\[
\mathbf{u}(0) = \mathbf{u}_0 \quad \text{and} \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad (10)
\]

where for simplicity, we have set \( \chi = 0 \) in (7) concerning the boundary conditions.
3.2. Norms and operators. We define the following bilinear forms and operators:

\[ A : X \to X' , \quad \langle Au, w \rangle_{X' \times X} = a(u, w) := \int_{\Omega} \varepsilon(u) : C : \varepsilon(w) \, dx , \]
\[ B : X \to Q' , \quad \langle Bu, p \rangle_{Q' \times Q} = b(u, p) := -\int_{\Omega} p \text{div} \, u \, dx , \]
\[ B' : Q \to X' , \quad \langle B'p, u \rangle_{X' \times X} = b(u, p) , \]
\[ C_\kappa : Q \to Q' , \quad \langle C_\kappa p, q \rangle_{Q' \times Q} = c_\kappa(p, q) := \int_{\Omega} (\kappa \nabla p) \cdot \nabla q \, dx , \]
\[ M : H \to H' , \quad \langle Mu, w \rangle_{H' \times H} = m(u, w) := \int_{\Omega} \rho u \cdot w \, dx . \]

The bilinear form \( a(\cdot, \cdot) \) defines an inner product on \( X \) due to Korn’s second inequality. Since \( a(\cdot, \cdot) \) is symmetric, the operator \( A \) is self-adjoint. We define a norm on \( X \) by setting

\[ \|u\|_X = \sqrt{a(u, u)} . \]

Similarly, the operator \( C_\kappa \) is self-adjoint and defines an inner product on \( Q \), but the associated norm is not appropriate in our context since this would result in parameter dependent estimates. Therefore, we define the parameter dependent norm

\[ \|p\|_Q = \sqrt{\|p\|_K^2 + \|p\|_S^2} , \]

with \( \|p\|_K = \sqrt{c_\kappa(p, p)} \) and \( \|p\|_S = \|A^{-1}B'p\|_X \), which now allows for the robust estimate

\[ \sup_{0 \neq u \in X} \frac{b(u, p)}{\|u\|_X} = \|B'p\|_{X'} = \|A^{-1}B'p\|_X \leq \|p\|_Q . \]

Finally, we use the weighted norm \( \|u\|_H = \sqrt{m(u, u)} \) on \( H \). Note that due to the Poincaré-Friedrichs inequality, for \( u \in X \), we can estimate

\[ \|u\|_H \leq C_\rho \|u\|_X , \]

where the constant \( C_\rho \) is only depending on the material parameters \( \rho, \mu, \lambda \) and the domain \( \Omega \).

3.3. Weak formulation. The finite element discretization of (9) is based on the weak formulation: find \( u \in C^2(0, T; V) \) and \( p(t) \in Q \) such that

\[ m(\ddot{u}(t), w) + a(u(t), w) + b(w, p(t)) = m(b(t), w) \quad w \in X , \quad (13a) \]
\[ b(\dot{u}(t), q) - c_\kappa(p(t), q) = 0 \quad q \in Q . \quad (13b) \]

Using the introduced notation, this can be rewritten as an operator equation in \( X' \times Q' \):

\[ M \ddot{u}(t) + Au(t) + B'p(t) = \ell(t) , \quad (14a) \]
\[ B\dot{u}(t) - C_\kappa p(t) = 0 , \quad (14b) \]

where \( \ell(t) \in X' \) is defined by \( \langle \ell(t), w \rangle = \int_{\Omega} \rho b(t) \cdot w \, dx . \)

3.4. Existence and uniqueness of a solution. Eliminating the pressure in (14) by the second equation gives \( p(t) = C_\kappa^{-1}B\ddot{u}(t) \), and substitution into the first equation yields the wave equation with damping

\[ M\ddot{u}(t) + D_\kappa \ddot{u}(t) + Au(t) = \ell(t) , \quad (15) \]

where \( D_\kappa = B'C_\kappa^{-1}B : X \to X' \). Since \( C_\kappa \) is monotone, so is the inverse \( C_\kappa^{-1} \) and consequently, also \( D_\kappa \) is monotone: \( \langle D_\kappa v, v \rangle \geq 0 \). Standard analysis applies to this problem (see, e. g., [33, Prop. I.6.1] for a proof in a modified setting with homogeneous right-hand side).

**Theorem 1.** For \( \ell \in C^0(0, T; X') \) the second order differential equation (15) with initial values (10) has a unique solution \( u \in C^2(0, T; H) \cap C^1(0, T; X) \).
As a consequence, we obtain \( p \in C^1(0,T;\mathbb{Q}) \). In case of smooth right-hand sides and initial values, higher regularity can be studied, cf. [16, Sect. 7.2.3] or [27] for an extended model with nonlinear damping and for a precise statement of the required assumptions on the data.

Note that the operator \( D_\kappa \) is not bounded independently of \( \kappa \), so that the pressure elimination is not suitable for a robust discretization.

### 3.5. Monotonicity

Introducing the velocity \( \nu(t) = \dot{u}(t) \) we consider an extended system. We define the product space

\[
\mathcal{W} = X \times X \times Q
\]

and the operators \( A,B: \mathcal{W} \to \mathcal{W}' \) by \( A[u,v,p] = (-Av, Au + B'v, -Bv + C_\kappa p) \) and by \( B[u,v,p] = (Au, Mv, 0) \), and the linear form \( \mathcal{L} \in \mathcal{W}' \) by \( \mathcal{L}(t)[u,v,p] = \langle \ell(t), v \rangle \).

Together with \( Av(t) = A\dot{u}(t) \) this rewrites (14) as an implicit evolution equation in \( \mathcal{W}' \)

\[
\partial_t (B w(t)) + A w(t) = \mathcal{L}(t), \quad Bw(0) = (Au_0, Mv_0, 0),
\]

for \( w(t) := (u(t), v(t), p(t)) \).

**Lemma 2.** The operators \( A \) and \( B \) are monotone.

**Proof.** This follows from \( \langle Bw, w \rangle = a(u,u) + \langle v,v \rangle_H \geq 0 \) and

\[
\langle Aw, w \rangle = c_\kappa(p,p) - b(v,v) + b(v,p) + a(u,v) - a(v,u) = c_\kappa(p,p) \geq 0,
\]

due to the symmetry of \( a(\cdot,\cdot) \) and the positivity of \( c_\kappa(\cdot,\cdot) \).

Note that \( B \) is singular, so that this has the structure of a differential algebraic equation. Thus, the application of semi-group theory requires a suitable factorization technique [33, Ch. IV]. This technique is applied in [35], where existence and uniqueness of a closely related system is investigated. In the following, monotonicity plays a major role in the estimates.

### 3.6. Energy estimates

The kinetic and potential energy of the wave equation

\[
\mathcal{E}_{\text{wave}}(u,v) = \frac{1}{2}m(v,v) + \frac{1}{2}a(u,u)
\]

is extended by the damping term, which defines the free energy by

\[
\mathcal{E}(u,p,t) = \mathcal{E}_{\text{wave}}(u(t), \dot{u}(t)) + \int_0^t c_\kappa(p(s), p(s)) \, ds,
\]

and the external work is given by

\[
\mathcal{E}_{\text{ext}}(v,t) = \int_0^t \langle \ell(s), v(s) \rangle \, ds.
\]

The initial energy is denoted by \( \mathcal{E}_0 = \mathcal{E}_{\text{wave}}(u_0, v_0) \).

**Theorem 3.** Assume \( b \in L^2(0,T;H) \), and let \( (u,p) \in H^1(0,T;X) \times L^2(0,T;\mathbb{Q}) \) be a solution of (13) for initial values \( (u_0,v_0) \in X \times X \). Then we have energy conservation in the form

\[
\mathcal{E}(u,p,t) = \mathcal{E}_0 + \mathcal{E}_{\text{ext}}(\dot{u},t), \quad t \in [0,T],
\]

(17a)

and an a priori estimate

\[
\mathcal{E}(u,p,t) \leq C(T) \left( \mathcal{E}_0 + \| b \|_{L^2(0,T;H)}^2 \right),
\]

(17b)

where \( C(T) \) is a constant only depending on time \( T \).
PROOF. Inserting \( w = \dot{u}(t) \) and \( q = p(t) \) in (13) yields \( b(\dot{u}(t), p(t)) = c_\kappa(p(t), p(t)) \) and
\[
m(\dot{u}(t), \ddot{u}(t)) + a(u(t), \dot{u}(t)) + c_\kappa(p(t), p(t)) = \langle \ell(t), \dot{u}(t) \rangle.
\]
This gives (17a) by
\[
\mathcal{E}_{\text{wave}}(u(t), v(t)) - \mathcal{E}_{\text{wave}}(u(0), v(0)) = \int_0^t \left( m(\dot{u}(s), \ddot{u}(s)) + a(u(s), \dot{u}(s)) \right) ds
= \int_0^t \left( \langle \ell(s), \dot{u}(s) \rangle - c_\kappa(p(s), p(s)) \right) ds.
\]
We obtain from Young’s inequality
\[
\mathcal{E}_{\text{ext}}(\dot{u}, t) = \int_0^t m(b(s), v(s)) ds \leq \int_0^t \| b(s) \|_H \| \dot{u}(s) \|_H ds
\leq \frac{1}{2} \int_0^t \| b(s) \|_H^2 ds + \frac{1}{2} \int_0^t \| \dot{u}(s) \|_H^2 ds,
\]
which gives
\[
\mathcal{E}(u, p, t) \leq \mathcal{E}_0 + \frac{1}{2} \int_0^t \| b(s) \|_H^2 ds + \int_0^t \mathcal{E}(u, p, s) ds.
\]
Then, (17b) is directly obtained from Gronwall’s Lemma [20, Lem. A.4.12]. □

As an immediate consequence in case of vanishing external work (\( \ell \equiv 0 \)), it follows that energy is conserved (i.e., \( \mathcal{E}(u, p, t) \equiv \mathcal{E}_0 \)), and the wave energy is dissipative in the form
\[
\frac{\partial}{\partial t} \mathcal{E}_{\text{wave}}(u(t), \dot{u}(t)) = -c_\kappa(p(t), p(t)) \leq 0.
\]

3.7. Lagrange principle. Finally, we introduce the corresponding Lagrange principle. For the generalized coordinates \( q = (u, p) \) define the Lagrangian by
\[
L(q, \dot{q}, t) = \frac{1}{2} \langle M \dot{u}(t), \dot{u}(t) \rangle - \frac{1}{2} \langle A u(t), u(t) \rangle + \int_0^t \langle C_\kappa p(s), p(s) \rangle ds - \langle B u(t), p(t) \rangle + \langle \ell(t), \dot{u}(t) \rangle.
\]
The corresponding Euler-Lagrange equation (obtained by the first variation of the action integral \( \frac{\partial}{\partial \ell} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} \)) yields the equation (14) in integrated form
\[
M \ddot{u}(t) + A u(t) + B' p(t) = \ell(t),
\]
\[
B u(t) - \int_0^t C_\kappa p(s) ds = 0.
\]

4. Discretization in time

In many cases physical restrictions require very small time steps for hyperbolic equations, and then explicit time stepping methods a favorable. Here, we consider applications in solid mechanics where in particular stability requirements would lead—in case of fine meshes—to unrealistic small time steps. Thus, one aims for time discretizations which are unconditionally stable. This can be easily obtained by enlarging the system by an equation for the velocity and the application of stable implicit Runge-Kutta schemes. However, the realization of such schemes is numerically expensive, and therefore in many cases Newmark schemes are applied: their realization has the same structure as implicit schemes for the quasi-static case, and it is quite easy to extend quasi-static applications to the full dynamic problem.

Of course, the implicit Euler schemes is also a stable scheme and could be applied alternatively. Nevertheless, since it is only of first order and not energy conserving (in case of no damping), in most applications the Newmark scheme is preferred.
4.1. The Newmark scheme. We consider the Newmark discretization in time for the system (13), see [28, Ch. 6.5]. Therefore, let \( t_0 < t_1 < t_2 < \cdots < t_N = T \) be a time series, and let \( \Delta t_n = t_n - t_{n-1} \) be the time increment. Set \( \ell^n = \ell(t_n) \).

Starting with \( u^0 = u_0 \) and \( v^0 = v_0 \) we compute for \( n = 0, 1, 2, \ldots, N \) the acceleration vector \( a^n \), the velocity vector \( v^n \), the displacement vector \( u^n \), and the pressure \( p^n \) satisfying

\[
m(a^n, w) + a(u^n, w) + b(w, p^n) = \langle \ell^n, w \rangle , \quad w \in X ,
\]

\[
b(v^n, q) - c_e(p^n, q) = 0 , \quad q \in Q ,
\]

and for \( n > 0 \)

\[
v^n = v^{n-1} + \frac{\Delta t_n}{2} (a^n + a^{n-1}) ,
\]

\[
u^n = u^{n-1} + \Delta t_n v^{n-1} + \frac{(\Delta t_n)^2}{4} (a^n + a^{n-1}) .
\]

Algorithmically, one can solve a saddle point problem for \((u^n, p^n)\)

\[
\frac{4}{(\Delta t_n)^2} m(u^n, w) + a(u^n, w) + b(w, p^n) = \langle r^n, w \rangle , \quad w \in X ,
\]

\[
b(u^n, q) - \frac{\Delta t_n}{2} c_e(p^n, q) = \langle r^n, q \rangle , \quad q \in Q ,
\]

with

\[\langle r^n, w \rangle = \langle \ell^n, w \rangle + \frac{4}{(\Delta t_n)^2} m(u^{n-1}, w) + \frac{4}{\Delta t_n} m(v^{n-1}, w) + m(a^{n-1}, w) , \quad w \in X ,\]

\[\langle r^n, q \rangle = b(u^{n-1}, q) + \frac{\Delta t_n}{2} b(v^{n-1}, q) , \quad q \in Q ,\]

and then recover \( v^n \) and \( a^n \) from

\[
v^n = \frac{2}{\Delta t_n} (u^n - u^{n-1}) - v^{n-1} ,
\]

\[
a^n = \frac{4}{(\Delta t_n)^2} (u^n - u^{n-1}) - \frac{4}{\Delta t_n} v^{n-1} - a^{n-1} .
\]

4.2. The Newmark discretization as a Nyström method. For the system

\[
\dot{u}(t) = v(t) , \quad \dot{v}(t) = a(t, u, \dot{u}) ,
\]

an \( s \)-stage Nyström method is given by

\[
k_i = a \left[ t_{n-1} + c_i \Delta t_n , \; u^{n-1} + c_i \Delta t_n v^{n-1} + (\Delta t_n)^2 \sum_{j=1}^s \overline{a}_{ij} k_j , \; v^{n-1} + \Delta t_n \sum_{j=1}^s \overline{a}_{ij} k_j \right] ,
\]

\[
u^n = u^{n-1} + \Delta t_n v^{n-1} + (\Delta t_n)^2 \sum_{i=1}^s \overline{b}_i k_i , \quad v^n = v^{n-1} + \Delta t_n \sum_{i=1}^s \overline{b}_i k_i .
\]

Following [18, Ch. II.14], a Nyström method is equivalent to a Runge-Kutta method, if the coefficients \( \overline{a}_{ij} \) and \( \overline{b}_i \) fulfill the relations

\[
\overline{a}_{ij} = \sum_{k=1}^s a_{ik} a_{kj} , \quad \overline{b}_i = \sum_{k=1}^s b_k a_{ki} .
\]

(24)

The Newmark method as it is defined above is a Nyström method \((s = 2\) stages\) with

\[
c = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \quad \overline{a} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} , \quad a = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} , \quad \overline{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \quad b = \begin{bmatrix} 2 \\ 2 \end{bmatrix} .
\]

A short computation evaluating (24) shows that this corresponds to the trapezoidal rule. This equivalence is useful for characterizing the Newmark method since all properties can be traced back to the trapezoidal rule. In particular, we can directly conclude that the Newmark method is A-stable and second order accurate.
The trapezoidal rule is symmetric in the sense of [17, Ch. 5], since the coefficients \( a_{ij}, b_i \) fulfill
\[
a_{s+1-i,s+1-j} + a_{ij} = b_j \quad \text{for all } 1 \leq i, j \leq s.
\]
Moreover, in the context of linear problems, the concepts of symmetry and symplecticity coincide [17, Ch. VI.4.2], and hence, quadratic invariants are conserved. In particular—in regard to the linear dynamic consolidation problem—this implies the energy conservation by the Newmark method. This fact is also reflected in the close relationship of the trapezoidal rule with the midpoint rule. For linear problems, both methods are equivalent, and since the midpoint rule is symplectic, so is the trapezoidal rule for linear problems. The trapezoidal rule is also referred to as conjugate symplectic to the midpoint rule [31, Ch. 14.3], since it is the result of a change of variables in the midpoint rule.

**Remark 4.** The general form of the Newmark method, as given in [28, Ch. 6.5], is parameterized by \( \beta \) and \( \gamma \) such that
\[
\begin{align*}
v^n &= v^{n-1} + \Delta t_n \left( \gamma a^n + (1 - \gamma) a^{n-1} \right), \\
u^n &= u^{n-1} + \Delta t_n v^{n-1} + (\Delta t_n)^2 \left( \beta a^n + \left( \frac{1}{2} - \beta \right) a^{n-1} \right).
\end{align*}
\]
The equivalent Nyström method is given by
\[
c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Xi = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} - \beta & \beta \end{bmatrix}, \quad a = \begin{bmatrix} 0 & 0 \\ 1 - \gamma & \gamma \end{bmatrix}, \quad \Theta = \begin{bmatrix} 1 - \beta \\ \gamma \end{bmatrix}.
\]
Suitable combinations of the parameters allow for controllable stability properties, and certain methods are retained in the general formulation, e.g. for \( \gamma = \frac{1}{2} \) and \( \beta = 0 \), the leap-frog (or Störmer-Verlet) algorithm is recovered. Since our fully discrete estimate in the next section is restricted to the basic Newmark scheme (\( \gamma = \frac{1}{2}, \beta = \frac{1}{2} \)), we do not analyse the variants here.

**Remark 5.** In the engineering literature, the Newmark method can be traced back to [25]. Based on the Newmark method, further extensions were proposed: the HHT-\( \alpha \) method of Hilber et. al. [19] and the WBZ-\( \alpha \) method of Wood et. al. [39] were combined to the generalized-\( \alpha \) method by Chung and Hulbert [10], yielding a four parameter method allowing for adjustable numerical dissipation while retaining second order accuracy and unconditional stability in the linear regime. As special cases, the trapezoidal rule and the midpoint rule are recovered.

### 5. Discretization in space

Since we aim for robust estimates with respect to \( \kappa \), a full control of the acceleration and the pressure is required. This is obtained by employing an inf-sup stable finite element discretization. Then, in order to prepare the structure of the proof for the full estimate in the next section, we consider a semidiscrete estimate.

Note that for the wave equation both, semi-discrete and fully discrete estimates for the displacements, are considered in [22, Th. 13.2], but for a robust estimate with respect to \( \kappa \) (based on the inf-sup stability) substantial extensions are required.

#### 5.1. The elliptic projection in space

Let \( h \in (0, h_0) \) be a discretization parameter and let \( X_h \times Q_h \subset X \times Q \) be a stable finite element discretization such that
\[
\sup_{u_h \in X_h} \frac{b(u_h, p_h)}{\|u_h\|_X} \geq \beta \sup_{u \in X} \frac{b(u, p_h)}{\|u\|_X}, \quad p_h \in Q_h,
\]
where \( \beta \) is independent of the discretization parameter \( h \). In particular, (12) implies
\[
\sup_{u_h \in X_h} \frac{b(u_h, p_h)}{\|u_h\|_X} \leq \frac{1}{\beta} \|p_h\|_Q, \quad p_h \in Q_h.
\]
For the discrete system, standard theory for linear ODE’s apply, see [1, Th. II.9.5]:
THEOREM 6. For $\ell \in C^k(0; T; X')$ the second order semidiscrete equation

$$\begin{align*}
m(\tilde{u}_h(t), w_h) + a(u_h(t), w_h) + b(w_h, p_h(t)) &= \langle \ell(t), w_h \rangle \quad w_h \in X_h, \\
b(\tilde{u}(t), q_h) - c_\kappa(p_h(t), q_h) &= 0 \quad q_h \in Q_h,
\end{align*}$$

with initial values

$$\begin{align*}
(u_h(0), \dot{u}_h(0)) &= (u_{h,0}, v_{h,0}) \in X_h \times X_h,
\end{align*}$$

has a unique solution $(u_h, p_h) \in C^{k+2}(0, T; X_h) \times C^{k+1}(0, T; Q_h)$.

For the semidiscrete analysis we introduce a coupled elliptic projection

$$(I_h, J_h) : X \times Q \rightarrow X_h \times Q_h$$

by solving the saddle point problem

$$\begin{align*}
a(I_h(u, p), w_h) + b(w_h, J_h(u, p)) &= a(u, w_h) + b(w_h, p) \quad w_h \in X_h, \\
b(I_h(u, p), q_h) - c_\kappa(J_h(u, p), q_h) &= b(u, q_h) - c_\kappa(p, q_h) \quad q_h \in Q_h.
\end{align*}$$

Elaborate stability investigations related with such saddle point problems have been performed in the context of penalized saddle point problems. In particular, we obtain quasi-optimal estimates uniformly in $\kappa$, see [3, 8].

THEOREM 7. A constant $C > 0$ depending only on $\beta$ exists such that

$$\|u - I_h(u, p)\|_X + \|p - J_h(u, p)\|_Q \leq C \inf_{(w_h, q_h) \in X_h \times Q_h} \left( \|u - w_h\|_X + \|p - q_h\|_Q \right).$$

Moreover, if $u \in H^2(\Omega, \mathbb{R}^3)$ and $p \in H^2(\Omega)$, and if the mesh is sufficiently regular, we have

$$\|u - I_h(u, p)\|_X + \|p - J_h(u, p)\|_Q \leq C(u, p) h,$$

with $C(u, p)$ independent of $\kappa$.

5.2. A semidiscrete estimate in space. The elliptic projection allows for decompositions

$$\begin{align*}
u(t) - u_h(t) &= \varphi(u(t) + \theta u(t), \varphi_u(t) = u(t) - I_h(u(t), p(t)), \theta u(t) = I_h(u(t), p(t)) - u_h(t), \\
p(t) - p_h(t) &= \varphi_p(t) + \theta_p(t), \quad \varphi_p(t) = p(t) - J_h(u(t), p(t)), \theta_p(t) = J_h(u(t), p(t)) - p_h(t),
\end{align*}$$

where first terms will be estimated by Theorem 7; the second term (in discrete spaces) is estimated in Theorem 8 below.

Temporal derivatives are defined accordingly, e. g., $\dot{\theta}_u(t) = I_h(\dot{u}(t), \dot{p}(t)) - \dot{u}_h(t)$.

THEOREM 8. If $u \in H^3(0, T; X)$ and $p \in H^2(0, T; Q)$, the estimate

$$\begin{align*}
\|\theta_p(t)\|_Q^2 + \|\dot{\theta}_u(t)\|_H^2 + \|\dot{\theta}_u(t)\|_X^2 + \|\theta u(t)\|_X^2 &\lesssim \|\dot{\varphi}_u(0)\|_H^2 + \|\dot{\varphi}_u(t)\|_X^2 + \|\theta u(0)\|_X^2 + \|\theta u(0)\|_Q^2 + \|\varphi_p(t)\|_Q^2 + \|\varphi u\|_H^3(0, t; X),
\end{align*}$$

holds, where the constant contained in $\lesssim$ is independent of $\kappa$.

PROOF. The construction of the projection implies

$$\begin{align*}
a(\varphi_u(t), w_h) + b(w_h, \varphi_p(t)) &= 0, \quad w_h \in X_h, \\
b(\varphi_u(t), q_h) - c_\kappa(\varphi_p(t), q_h) &= 0, \quad q_h \in Q_h,
\end{align*}$$

and $b(\dot{\varphi}_u(t), q_h) - c_\kappa(\theta_p(t), q_h) = 0$. Subtracting (13) and (27) and inserting (29) gives

$$\begin{align*}
m(\tilde{\theta}_u(t), w_h) + a(\theta_u(t), w_h) + b(w_h, \theta_p(t)) + m(\tilde{\varphi}_u(t), w_h) &= 0 \quad w_h \in X_h, \\
b(\tilde{\theta}_u(t), q_h) - c_\kappa(\theta_p(t), q_h) + b(\varphi_u(t) - \theta u(t), q_h) &= 0 \quad q_h \in Q_h.
\end{align*}$$

Choosing $w_h = \tilde{\theta}_u(t)$ and $q_h = \theta_p(t)$ in (30) yields

$$\begin{align*}
m(\tilde{\varphi}_u(t), \tilde{\theta}_u(t)) + a(\theta u(t), \tilde{\theta}_u(t)) + c_\kappa(\theta_p(t), \theta_p(t)) &= -m(\tilde{\varphi}_u(t), \dot{\theta}_u(t)) + b(\tilde{\theta}_u(t) - \varphi_u(t), \theta_p(t)),
\end{align*}$$
and integrating results in
\[ \| \ddot{\theta}_u(t) \|^2_{\mathcal{H}} + \| \theta_u(t) \|^2_x + 2 \int_0^t \| \theta_p(s) \|^2_x \, ds \]
\[ = \| \ddot{\theta}_u(0) \|^2_{\mathcal{H}} + \| \theta_u(0) \|^2_x + 2 \int_0^t b(\dot{\theta}_u(s) - \theta_u(s), \theta_p(s)) \, ds - 2 \int_0^t m(\ddot{\theta}_u(s), \dot{\theta}_u(s)) \, ds \]
\[ \leq C_1(u, p) + \int_0^t \| \theta_p(s) \|^2_x \, ds + \int_0^t \| \dot{\theta}_u(s) \|^2_{\mathcal{H}} \, ds, \] (31)
with
\[ C_1(u, p) = \| \ddot{\theta}_u(0) \|^2_{\mathcal{H}} + \| \theta_u(0) \|^2_x + \int_0^t \| \dot{\theta}_u(s) - \theta_u(s) \|^2_x \, ds + \int_0^t \| \ddot{\theta}_u(s) \|^2_{\mathcal{H}} \, ds. \]

Differentiating (29) and (30) in time analogously yields
\[ \| \ddot{\theta}_u(t) \|^2_{\mathcal{H}} + \| \theta_u(t) \|^2_x + 2 \int_0^t \| \theta_p(s) \|^2_x \, ds \]
\[ = \| \ddot{\theta}_u(0) \|^2_{\mathcal{H}} + \| \theta_u(0) \|^2_x + 2 \int_0^t b(\ddot{\theta}_u(s) - \dot{\theta}_u(s), \theta_p(s)) \, ds - 2 \int_0^t m(\dddot{\theta}_u(s), \dot{\theta}_u(s)) \, ds. \]
Using integration by parts and Young’s inequality, we insert the estimate
\[ \int_0^t b(\ddot{\theta}_u(s) - \dot{\theta}_u(s), \theta_p(s)) \, ds \]
\[ = b(\ddot{\theta}_u(t) - \dot{\theta}_u(t), \theta_p(t)) - b(\ddot{\theta}_u(0) - \dot{\theta}_u(0), \theta_p(0)) - \int_0^t b(\dddot{\theta}_u(s), \theta_p(s)) \, ds \]
\[ \leq \frac{1}{2 \eta} \| \ddot{\theta}_u(t) - \dot{\theta}_u(t) \|^2_x + \frac{\eta}{2} \| \theta_p(t) \|^2_x + \frac{1}{2} \| \ddot{\theta}_u(0) - \dot{\theta}_u(0) \|^2_x + \frac{1}{2} \| \theta_p(0) \|^2_x \]
\[ + \frac{1}{2} \int_0^t \| \dddot{\theta}_u(s) \|^2_x \, ds + \frac{1}{2} \int_0^t \| \theta_p(s) \|^2_x \, ds, \]
which gives together
\[ \| \ddot{\theta}_u(t) \|^2_{\mathcal{H}} + \| \theta_u(t) \|^2_x \leq \eta \| \theta_p(t) \|^2_x + C_2(u, p) + \int_0^t \| \theta_p(s) \|^2_x \, ds + \int_0^t \| \dot{\theta}_u(s) \|^2_{\mathcal{H}} \, ds, \] (32)
with
\[ C_2(u, p) = \| \ddot{\theta}_u(0) \|^2_{\mathcal{H}} + \| \theta_u(0) \|^2_x + \frac{1}{\eta} \| \ddot{\theta}_u(t) - \dot{\theta}_u(t) \|^2_x + \| \ddot{\theta}_u(0) - \dot{\theta}_u(0) \|^2_x + \| \theta_p(0) \|^2_x \]
\[ + \int_0^t \| \dddot{\theta}_u(s) \|^2_x \, ds + \int_0^t \| \dddot{\theta}_u(s) \|^2_{\mathcal{H}} \, ds. \]
Estimating \( b(w_{\mathcal{H}}, \theta_p(t)) \) in (30a) gives
\[ \| \theta_p(t) \|_{\mathcal{H}} \leq C_p \left( \| \ddot{\theta}_u(t) \|_{\mathcal{H}} + \| \dddot{\theta}_u(t) \|_{\mathcal{H}} \right) + \| \theta_u(t) \|_{x}, \] (33)
and choosing \( q_h = \theta_p(t) \) in (30b) results in
\[ \| \theta_p(t) \|^2_x \leq \| \ddot{\theta}_u(t) \|^2_x + \| \dddot{\theta}_u(t) - \dot{\theta}_u(t) \|^2_x + \| \theta_u(t) - \theta_u(t) \|^2_x. \] (34)
From (33) and (34) we obtain
\[ \| \theta_p(t) \|^2_{\mathcal{H}} \leq 3C_p^2 \left( \| \ddot{\theta}_u(t) \|^2_{\mathcal{H}} + \| \dddot{\theta}_u(t) \|^2_{\mathcal{H}} + 3\| \theta_u(t) \|^2_x + \| \dddot{\theta}_u(t) \|^2_x + \| \dddot{\theta}_u(t) - \dot{\theta}_u(t) \|^2_x \right), \]
and inserting (31) and (32) and choosing \( \eta > 0 \) sufficiently small combines to
\[ \| \theta_p(t) \|^2_{\mathcal{H}} + \| \ddot{\theta}_u(t) \|^2_{\mathcal{H}} + \| \theta_u(t) \|^2_x \]
\[ \leq C_3(u, p) + \int_0^t \left( \| \theta_p(s) \|^2_x + \| \ddot{\theta}_u(s) \|^2_{\mathcal{H}} \, ds + \| \dot{\theta}_u(s) \|^2_x \right) \, ds, \]
with
\[ C_3(u, p) = C_1(u, p) + C_2(u, p) + \| \ddot{\theta}_u(t) - \dot{\theta}_u(t) \|^2_x + \| \dddot{\theta}_u(t) \|^2_{\mathcal{H}}. \]
Now, we apply Gronwall’s lemma [20, Lem. A.4.12] and Sobolev’s embedding theorem [16, Th. 5.9.2]; this gives the assertion. \( \square \)
COROLLARY 9. If \( u \in H^3(0, T; H^2(\Omega, \mathbb{R}^3)) \) and \( p \in H^3(0, T; H^2(\Omega)) \), we have in case of exact initial values

\[
\|\tilde{u}(t) - \tilde{u}_h(t)\|_H + \|\dot{u}(t) - \dot{u}_h(t)\|_X + \|u(t) - u_h(t)\|_X + \|p(t) - p_h(t)\|_Q \lesssim C(u) \ h.
\]

PROOF. The desired estimate is obtained from

\[
\|\tilde{u}(t) - \tilde{u}_h(t)\|_H^2 + \|\dot{u}(t) - \dot{u}_h(t)\|_X^2 + \|u(t) - u_h(t)\|_X^2 + \|p(t) - p_h(t)\|_Q^2
\]

\[
\leq \|\tilde{\vartheta}_u(t)\|_H^2 + \|\dot{\vartheta}_u(t)\|_X^2 + \|\vartheta_u(t)\|_X^2 + \|\vartheta_p(t)\|_Q^2
\]

\[
+ \|\tilde{\vartheta}_u(t)\|_H^2 + \|\dot{\vartheta}_u(t)\|_X^2 + \|\vartheta_u(t)\|_X^2 + \|\vartheta_p(t)\|_Q^2
\]

and inserting Theorem 7 for the estimate of the interpolation error. \( \square \)

6. Convergence in space and time

6.1. The Newmark discretization as a finite difference method. For simplicity, we restrict ourselves to the case of uniform time step size \( \Delta t_n = \Delta t \).

Furthermore, let \( \Delta^2 w^n = w^n - w^{n-1} \) and \( \Delta w^n = \Delta(\Delta w^n) = w^n - 2w^{n-1} + w^{n-2} \) be the first and second finite difference, let \( \partial_{\Delta t} w^n = (\Delta t)^{-1} \Delta w^n \) and \( \partial_{\Delta t}^2 w^n = \partial_{\Delta t}(\partial_{\Delta t}^{-1} \Delta w^n) \) be the difference quotients in time, and we introduce the averaged values \( \{w^n\} = \frac{1}{2}(w^n + w^{n-1}) \) and \( \{\{w^n\}\} = \frac{1}{2}\{w^n\} + \{w^{n-1}\} \).

For time-continuous quantities, the above operators have to be interpreted as evaluations at times \( t = t_n \), e.g. \( \{u(t_n)\} = \frac{1}{2}(u(t_n) + u(t_{n-1})) \).

The fully discrete scheme is defined as follows: for given initial values \( u^n_0, v^n_0 \in \mathcal{X}_h \), compute approximations \( u^n_h, v^n_h, a^n_h \in \mathcal{X}_h \) and \( p^n_h \in Q_h \), satisfying

\[
m(a^n_h, w_h) + a(u^n_h, w_h) + b(w_h, p^n_h) = \{\ell^n, w_h\} \quad w_h \in \mathcal{X}_h, \tag{35a}
\]

\[
b(v^n_h, q_h) - c_n(p^n_h, q_h) = 0 \quad q_h \in Q_h, \tag{35b}
\]

for \( n = 0, 1, 2, \ldots, N \), and for \( n > 0 \) (according to (23))

\[
v^n_h = \frac{2}{\Delta t_n} \Delta u^n_h - v^{n-1}_h, \quad u^n_h = \frac{4}{(\Delta t_n)^2} \Delta u^n_h - \frac{4}{\Delta t_n} v^{n-1}_h - a^{n-1}_h, \tag{36}
\]

For \( n > 1 \), we observe the identities \( \{v^n_h\} = \partial_{\Delta t} u^n_h \) and \( \{\{a^n_h\}\} = \partial_{\Delta t}^2 u^n_h = \{\partial_{\Delta t} v^n_h\} \).

In analogy to Sect. 5.2 we define

\[
\vartheta_u^n := u(t_n) - I_h(u(t_n), p(t_n)) \in \mathcal{X}, \quad \vartheta_u^n := I_h(u(t_n), p(t_n)) - u^n_h \in \mathcal{X}_h, \tag{35b}
\]

\[
\vartheta_p^n := p(t_n) - J_h(u(t_n), p(t_n)) \in Q, \quad \vartheta_p^n := J_h(u(t_n), p(t_n)) - p^n_h \in Q_h, \tag{35b}
\]

and we set

\[
\xi_u^n := u(t_n) - u^n_h = \vartheta_u^n + \vartheta_{\Delta t} u^n, \quad \xi_{\Delta t} p^n := p(t_n) - p^n_h = \vartheta_p^n + \vartheta_{\Delta t} p^n.
\]

The corresponding quantities for the velocities and the acceleration are denoted by \( \vartheta_v^n, \vartheta_{\Delta t} v^n, \vartheta_v^n \) and \( \vartheta_a^n, \vartheta_{\Delta t} a^n, \vartheta_{\Delta t} a^n \), respectively.

The estimate in the fully discrete case follows the lines of the semi-discrete estimate, where the Gronwall Lemma is replaced by the discrete version [26, Lemma 1.4.2], and the integration by parts is also replaced by a discrete counterpart.

LEMMA 10. Assume \( C_0 > 0, \) and that the sequence \( \{a_n\} \) is non-negative. If the sequence \( \{\varphi_n\} \) satisfies

\[
\phi_0 \leq C_0 \text{ and } \phi_n \leq C_0 + \sum_{i=0}^{n-1} a_i \phi_i \text{ for } n \geq 1, \text{ then } \phi_n \leq C_0 \exp\left(\sum_{i=0}^{n-1} a_i\right) \text{ holds.}
\]

LEMMA 11. We have for an arbitrary bilinear form \( B(\cdot, \cdot) \) and sequences \( u^n, v^n \)

\[
B(\{u^n\}, \partial_{\Delta t} v^n) = -B(\partial_{\Delta t} u^n, \{v^n\}) + \partial_{\Delta t} B(u^n, v^n),
\]

If additionally, the bilinear form is symmetric we have

\[
2\Delta t B(\{u^n\}, \partial_{\Delta t} u^n) = B(u^n, u^n) - B(u^{n-1}, u^{n-1}). \tag{37}
\]
PROOF. It holds
\[
B\{u^n, \partial_\Delta t v^n\} + B(\partial_\Delta t u^n, \{v^n\}) = \frac{1}{2\Delta t} \left( B(u^n + u^{n-1}, v^n - v^{n-1}) + B(u^n - u^{n-1}, v^n + v^{n-1}) \right) = \frac{1}{2\Delta t} \left( 2B(u^n, v^n) - 2B(u^{n-1}, v^{n-1}) \right) = \partial_\Delta t B(u^n, v^n).
\]

The second statement immediately follows.

The error in space will be estimated in terms of \(e^n_u\).

**Lemma 12.** We have for \(n > 1\)
\[
\Delta t \|\partial_\Delta t e^n_u\|^2 \leq \|\hat{e}_u\|^2_{L^2(t_n-1, t_n, X)} \leq 2\Delta t \|\hat{e}_u\|^2_{L^\infty(t_n-2, t_n, X)},
\]
and for \(n > 2\)
\[
\Delta t \|\partial^2_\Delta t e^n_u\|^2 \leq \|\tilde{e}_u\|^2_{L^2(t_n-2, t_n, X)} \leq 3\Delta t \|\hat{e}_u\|^2_{L^\infty(t_n-2, t_n, X)},
\]
Moreover, it holds for \(n > 1\)
\[
\Delta t \|\{e^n_u\}\|^2 \leq \|\hat{e}_u\|^2_{L^2(t_n-2, t_n, X)} + (\Delta t)^4 \|\hat{e}_u\|^2_{L^2(t_n-2, t_n, X)}.
\]

**Proof.** We only show (38a) since the others follow similarly. Taylor expansions gives
\[
e^n_u = e^{n-1}_u + \int_{t_{n-1}}^{t_n} \hat{e}_u(s) \, ds, \quad \text{and} \quad e^{n-2}_u = e^{n-1}_u - \int_{t_{n-2}}^{t_{n-1}} \hat{e}_u(s) \, ds,
\]
and hence \(\partial_\Delta t e^n_u = (2\Delta t)^{-1} \int_{t_{n-2}}^{t_n} \hat{e}_u(s) \, ds\). Applying \(\|\cdot\|^2_X\) and Hölder’s inequality gives
\[
\|\partial_\Delta t e^n_u\|^2_X = (2\Delta t)^{-2} \left\| \int_{t_{n-2}}^{t_n} \hat{e}_u(s) \, ds \right\|^2_X \\
\leq (2\Delta t)^{-2} \int_{t_{n-2}}^{t_n} 1^2 \, ds \int_{t_{n-2}}^{t_n} \|\hat{e}_u(s)\|^2_X \, ds = (2\Delta t)^{-1} \|\hat{e}_u\|^2_{L^2(t_{n-2}, t_n, X)},
\]
which is the first assertion. The estimate (38e) is achieved by using the error estimate of the trapezoidal quadrature rule.

Summing up suitable quantities (as they will be used in the proof of the following theorem), for
\[
S^k := \|\partial^2_\Delta t e^k_u\|^2_H + \|\partial_\Delta t e^k_u\|^2_X + \|\{e^k_u\}\|^2_X + \|\partial^2_\Delta t e^{k-1}_u\|^2_H + \|\partial_\Delta t e^{k-1}_u\|^2_X + \|\partial^3_\Delta t e^k_u\|^2_X + \|\partial^3_\Delta t e^{k-1}_u\|^2_X
\]
this Lemma immediately gives the bound
\[
\Delta t \sum_{k=3}^{n} S^k \leq (1 + (\Delta t)^4) \|\hat{e}_u\|^2_{H^3(0, t_n; X)}.
\]

The error in time is estimated by comparison with finite differences in the following Lemma.

**Lemma 13.** We have for \(n > 1\)
\[
\{\xi^n_o\} = \partial^3_\Delta t \xi^n_u + \delta^n_1, \quad with \quad \Delta t \|\delta^n_1\|^2_H \leq C(\Delta t)^4 \|\hat{e}^4_u\|^2_{L^2(t_n-2, t_n, H)},
\]
\[
\{\xi^n_v\} = \partial_\Delta t \xi^n_u + \delta^n_2, \quad with \quad \Delta t \|\delta^n_2\|^2_X \leq C(\Delta t)^4 \|\hat{e}^3_u\|^2_{L^2(t_n-2, t_n, X)},
\]
\[
\{\partial_\Delta t \xi^n_v\} = \partial^2_\Delta t \xi^n_u + \delta^n_3, \quad with \quad \|\delta^n_3\|^2_X \leq C(\Delta t)^4 \|\hat{e}^3_u\|^2_{L^\infty(t_n-2, t_n, X)}.
\]
and for \( n > 2 \)

\[
\begin{align*}
\{\partial_{\Delta t} \Theta^n_w\} &= \partial_{\Delta t}^3 \Theta^n_w + \delta_4^n \quad \text{with } \Delta t \| \delta_4^n \|_H^2 \leq C(\Delta t)^4 \| \partial_1^5 u \|_{L_2(t_{n-3}, t_n; H)}^2, \quad (40d) \\
\{\partial_{\Delta t}^2 \Theta^n_v\} &= \partial_{\Delta t}^3 \Theta^n_v + \delta_5^n \quad \text{with } \Delta t \| \delta_5^n \|_X^2 \leq C(\Delta t)^4 \| \partial_1^5 u \|_{L_2(t_{n-3}, t_n; X)}^2. \quad (40e)
\end{align*}
\]

**PROOF.** Again, we only show the first estimate since the others may be obtained likewise. By Taylor expansion, we find that

\[
\{\tilde{u}(t_n)\} = \partial_{\Delta t}^2 u(t_n) + \int_{t_{n-1}}^{t_n} (t_n - s) \left( \frac{1}{4} - \frac{1}{6(\Delta t)^2}(t_n - s)^2 \right) \partial_1^4 u(s) \, ds \\
+ \int_{t_{n-2}}^{t_{n-1}} (s - t_{n-2}) \left( \frac{1}{4} - \frac{1}{6(\Delta t)^2}(s - t_{n-2})^2 \right) \partial_1^4 u(s) \, ds,
\]

and inserting \( \{a^n_h\} = \partial_{\Delta t} u_h^n \) gives together with Hölder’s inequality

\[
\| \delta_4^n \|_H^2 = \| \{\tilde{u}(t_n)\} - \partial_{\Delta t}^2 u(t_n) \|_H^2 \\
\leq 2 \left\| \int_{t_{n-1}}^{t_n} (t_n - s) \left( \frac{1}{4} - \frac{1}{6(\Delta t)^2}(t_n - s)^2 \right) \partial_1^4 u(s) \, ds \right\|_H^2 \\
+ 2 \left\| \int_{t_{n-2}}^{t_{n-1}} (s - t_{n-2}) \left( \frac{1}{4} - \frac{1}{6(\Delta t)^2}(s - t_{n-2})^2 \right) \partial_1^4 u(s) \, ds \right\|_H^2 \\
\leq 2 \int_{t_{n-1}}^{t_n} \left( t_n - s \right) \left( \frac{1}{4} - \frac{1}{6(\Delta t)^2}(t_n - s)^2 \right)^2 \, ds \int_{t_{n-1}}^{t_n} \| \partial_1^4 u(s) \|_H^2 \, ds \\
+ 2 \int_{t_{n-2}}^{t_{n-1}} \left( s - t_{n-2} \right) \left( \frac{1}{4} - \frac{1}{6(\Delta t)^2}(s - t_{n-2})^2 \right)^2 \, ds \int_{t_{n-2}}^{t_{n-1}} \| \partial_1^4 u(s) \|_H^2 \, ds \\
= \frac{41}{2520} (\Delta t)^3 \| \partial_1^4 u \|_{L_2(t_{n-2}, t_n; H)}^2.
\]

\[ \square \]

Again, we summarize for later use: the term

\[
T^k := \| \delta_1^k \|_H^2 + \| \delta_2^k \|_X^2 + \| \delta_1^{k-1} \|_H^2 + \| \delta_2^{k-1} \|_X^2 + \| \delta_4^k \|_H^2 + \| \delta_5^k \|_X^2,
\]

can be estimated by

\[
\Delta t \sum_{k=3}^n T^k \lesssim (\Delta t)^4 \| u \|_{H^2(0, t_n; X)}^2.
\]

**6.2. A fully discrete estimate in time and space.** Having finished the preparations, we present a fully discrete estimate. As indicated, the approach is akin to the spatially discrete estimate of the last section.

**THEOREM 14.** If \( u \in H^3(0, T, X) \) and \( p \in H^3(0, T, Q) \), the estimate

\[
\| \partial_{\Delta t}^2 \Theta_w^n \|_H^2 + \| \{ \partial_{\Delta t} \Theta_v^n \} \|_X^2 + \| \{ \Theta_w^n \} \|_X^2 + \| \{ \Theta_v^n \} \|_Q^2 \\
\lesssim C(\| u_0 \|_{H^3} + \| p_0 \|_{H^3}) + (\Delta t)^4 \| u \|_{H^2(0, T, X)} + \| \Theta_w^n \|_{H^3(0, T, X)}
\]

holds with constants independent of \( \kappa, \Delta t, \) and \( h \).

**PROOF.** The construction of the projection (28a) implies

\[
\begin{align*}
a(\Theta_w^n, w_h) + b(w_h, \Theta_v^n) &= 0 \quad \text{for } \| w_h \|_{X_h}, \quad (42a) \\
b(g_w^n, q_h) - c_k(g_v^n, q_h) &= 0 \quad \text{for } \| q_h \|_{Q_h}, \quad (42b)
\end{align*}
\]

Subtracting (13) for \( t = t_n \) and inserting (35) gives

\[
\begin{align*}
m(\Theta_w^n, w_h) + a(\Theta_w^n, w_h) + b(w_h, \Theta_p^n) + m(g_w^n, w_h) &= 0, \quad \text{for } \| w_h \|_{X_h}, \quad (43a) \\
b(\Theta_v^n, q_h) - c_k(\Theta_p^n, q_h) + b(g_v^n - g_w^n, q_h) &= 0, \quad \text{for } \| q_h \|_{Q_h}. \quad (43b)
\end{align*}
\]
Averaging and inserting \(w_h = \{\partial_{\Delta t} \theta^0_u\}\) and \(q_h = \{\{\theta^0_p\}\}\) in (43), for \(n > 1\), together yields

\[
m(\{\theta^n_u\}, \{\partial_{\Delta t} \theta^n_u\}) + a(\{\theta^n_u\}, \{\partial_{\Delta t} \theta^n_u\}) + c_n(\{\theta^n_p\}, \{\theta^n_p\}) \\
= b(\{\theta^n_u\}, \{\theta^n_u\}) - \{\partial_{\Delta t} \theta^n_u\}, \{\theta^n_p\}) ,
\]

and inserting (40) gives

\[
m(\partial^2_{\Delta t} \theta^n_u, \partial_{\Delta t} \theta^n_u) + a(\{\theta^n_u\}, \partial_{\Delta t} \theta^n_u) + c_n(\{\theta^n_p\}, \{\theta^n_p\}) \\
= m(\partial^2_{\Delta t} \theta^n_u + \delta^n_1, \partial_{\Delta t} \theta^n_u) + b(\{\partial_{\Delta t} \theta^n_u\}, \{\theta^n_p\}) \\
\leq \|\partial^2_{\Delta t} \theta^n_u + \delta^n_1\|_H \|\partial_{\Delta t} \theta^n_u\|_H + \|\partial_{\Delta t} \theta^n_u\| + \|\theta^n_p\|_X \|\{\{\theta^n_p\}\}\|_S \\
\leq \frac{1}{2\eta_1} \left(\|\partial^2_{\Delta t} \theta^n_u + \delta^n_1\|_H^2 + \|\partial_{\Delta t} \theta^n_u\|_H^2 + \|\{\partial_{\Delta t} \theta^n_u\}\|_X^2 + \frac{\eta_1}{2} \left(\|\partial_{\Delta t} \theta^n_u\|_H^2 + \|\{\theta^n_p\}\|_S^2\right)\right) ,
\]

where \(\eta_1 \in (0, 1)\) will be fixed later. From (37) we obtain

\[
\|\partial_{\Delta t} \theta^n_u\|_H^2 + \|\|\{\theta^n_u\}\|_X^2 \\
= \|\partial_{\Delta t} \theta^n_{u-1}\|_H^2 + \|\{\theta^n_{u-1}\}\|_X^2 + 2\Delta t \left(m(\partial^2_{\Delta t} \theta^n_u, \partial_{\Delta t} \theta^n_u) + a(\{\theta^n_u\}, \partial_{\Delta t} \theta^n_u)\right) \\
\leq \|\partial_{\Delta t} \theta^n_{u-1}\|_H^2 + \|\theta^n_{u-1}\|_X^2 + \frac{\Delta t^2}{\eta_1} \left(\|\partial^2_{\Delta t} \theta^n_u + \delta^n_1\|_H^2 + \|\partial_{\Delta t} \theta^n_u\|_H^2 + \|\{\partial_{\Delta t} \theta^n_u\}\|_X^2 + \frac{\eta_1}{2} \left(\|\partial_{\Delta t} \theta^n_u\|_H^2 + \|\{\theta^n_p\}\|_S^2\right)\right) .
\]

Summing up and using (39) and (41) gives

\[
\|\partial_{\Delta t} \theta^n_u\|_H^2 + \|\|\{\theta^n_u\}\|_X^2 \\
\leq C_1 + \frac{\Delta t}{\eta_1} \sum_{k=3}^n (S^k + T^k) + \Delta t \sum_{k=2}^{n-1} \left(\|\{\partial_{\Delta t} \theta^n_u\}\|_H^2 + \|\{\theta^n_p\}\|_S^2\right) \\
+ \eta_1 \Delta t \left(\|\partial_{\Delta t} \theta^n_u\|_H^2 + \|\{\theta^n_p\}\|_S^2\right) , \quad \text{(44)}
\]

with \(C_1 = \|\partial_{\Delta t} \theta^1_u\|_H^2 + \|\{\theta^1_u\}\|_X^2\). Again using (43), we obtain for \(n > 2\)

\[
m(\{\partial_{\Delta t} \theta^n_u\}, w_h) + a(\{\partial_{\Delta t} \theta^n_u\}, w_h) + b(\{\theta^n_p\}, \{\partial_{\Delta t} \theta^n_u\}) \\
+ m(\{\partial_{\Delta t} \theta^n_u\}, \{\theta^n_p\}) = 0 ,
\]

\[
b(\{\partial_{\Delta t} \theta^n_u\}, q_h) - c_n(\{\partial_{\Delta t} \theta^n_p\}, q_h) + b(\{\partial_{\Delta t} \theta^n_u - \partial_{\Delta t} \theta^n_p\}, q_h) = 0 ,
\]

and inserting \(w_h = \{\partial_{\Delta t} \theta^n_u\}\) and \(q_h = \{\partial_{\Delta t} \theta^n_p\}\) gives

\[
m(\{\partial_{\Delta t} \theta^n_u\}, \{\partial_{\Delta t} \theta^n_u\}) + a(\{\partial_{\Delta t} \theta^n_u\}, \{\partial_{\Delta t} \theta^n_u\}) + b(\{\partial_{\Delta t} \theta^n_u\}, \{\partial_{\Delta t} \theta^n_p\}) = 0 ,
\]

\[
b(\{\partial_{\Delta t} \theta^n_u\}, \{\partial_{\Delta t} \theta^n_p\}) - c_n(\{\partial_{\Delta t} \theta^n_p\}, \{\partial_{\Delta t} \theta^n_p\}) + b(\{\partial_{\Delta t} \theta^n_u - \partial_{\Delta t} \theta^n_p\}, \{\partial_{\Delta t} \theta^n_p\}) = 0 ,
\]

and we obtain

\[
m(\{\partial_{\Delta t} \theta^n_u\}, \{\partial_{\Delta t} \theta^n_u\}) + a(\{\partial_{\Delta t} \theta^n_u\}, \{\partial_{\Delta t} \theta^n_u\}) + c_n(\{\partial_{\Delta t} \theta^n_p\}, \{\partial_{\Delta t} \theta^n_p\}) \\
= b(\partial_{\Delta t} \{\theta^n_u\} - \theta^n_u) - \{\partial_{\Delta t} \theta^n_u\} , \{\partial_{\Delta t} \theta^n_u\} ) .
\]

Inserting (40) and using Lemma 11 gives

\[
m(\partial^2_{\Delta t} \theta^n_u + \delta^n_1, \{\partial_{\Delta t} \theta^n_u\}) + a(\{\partial_{\Delta t} \theta^n_u\}, \{\partial_{\Delta t} \theta^n_u\}) + c_n(\{\partial_{\Delta t} \theta^n_p\}, \{\partial_{\Delta t} \theta^n_p\}) \\
= b(\partial_{\Delta t} \{\theta^n_u\} - \theta^n_u) - \{\partial_{\Delta t} \theta^n_u\} + \delta^n_1u + \{\theta^n_p\}) \\
- \partial_{\Delta t} b(\partial_{\Delta t} \{\theta^n_u\} - \theta^n_u) + \partial_{\Delta t} \theta^n_u + \{\theta^n_p\}) \\
= -b(\delta^n_1 + \partial_{\Delta t} \theta^n_u - \{\partial_{\Delta t} \theta^n_u\}, \{\theta^n_u\}) + \partial_{\Delta t} b(\delta^n_1 + \partial_{\Delta t} \theta^n_u - \{\partial_{\Delta t} \theta^n_u\}, \{\theta^n_p\}) .
\]
Now we set \( \eta^n_t = \partial^3_{\Delta t} \theta^n_u + \delta^n_t, \eta^n_2 = \delta^n_t + \partial^3_{\Delta t} \theta^n_u - \{ \partial^2_{\Delta t} \theta^n_u \}, \eta^n_3 = \delta^n_t + \partial^2_{\Delta t} \theta^n_u - \{ \partial_{\Delta t} \theta^n_u \} \). From (37) we obtain

\[
\| \partial^2_{\Delta t} \theta^n_u \|^2_{H^2} + \| \{ \partial_{\Delta t} \theta^n_u \} \|^2_X = \| \partial^2_{\Delta t} \theta^{n-1}_u \|^2_H + \| \{ \partial_{\Delta t} \theta^{n-1}_u \} \|^2_X \\
+ 2 \Delta t \left( m_{\partial^2_{\Delta t} \theta^n_u, \{ \partial_{\Delta t} \theta^n_u \}} + a(\{ \partial_{\Delta t} \theta^n_u \}, \partial^2_{\Delta t} \theta^n_u) \right) \\
\leq \| \partial^2_{\Delta t} \theta^{n-1}_u \|^2_H + \| \{ \partial_{\Delta t} \theta^{n-1}_u \} \|^2_X + 2 \Delta t \| \eta^n_t \|_H \| \{ \partial^2_{\Delta t} \theta^n_u \} \|_H \\
+ 2 \Delta t \| \eta^n_2 \|_X + 2 \Delta t \| \partial_{\Delta t} \theta^n_u \|_X \cdot
\]

Summing up and using \( \| \{ \theta^n_p \} \|_S \leq \frac{1}{2} \| \{ \theta^n_p \} \|_s + \frac{1}{2} \| \{ \theta^n_p \} \|_S \) gives

\[
\| \partial^2_{\Delta t} \theta^n_u \|^2_{H^2} + \| \{ \partial_{\Delta t} \theta^n_u \} \|^2_X \\
\leq \| \partial^2_{\Delta t} \theta^{n-1}_u \|^2_H + \| \{ \partial_{\Delta t} \theta^{n-1}_u \} \|^2_X + 2 \Delta t \sum_{k=3}^{n-1} \left( \| \eta^n_t \|_H \| \{ \partial^2_{\Delta t} \theta^n_u \} \|_H + \| \eta^n_2 \|_X \| \{ \theta^n_p \} \|_S \right) \\
+ 2 b(\eta^n_3, \{ \theta^n_p \}) - 2 b(\eta^n_3, \{ \theta^n_p \}) \\
\leq C_2 + \frac{\Delta t}{2} \sum_{k=3}^{n-1} (s^k + T^k) + \frac{1}{\eta^n_3} \| \eta^n_3 \|^2_X + 2 \Delta t \sum_{k=3}^{n-1} \left( \| \partial^2_{\Delta t} \theta^n_u \|^2_H + \| \{ \theta^n_p \} \|_S \right) \\
+ \eta^n_2 \Delta t \left( \| \partial^2_{\Delta t} \theta^n_u \|^2_H + \| \{ \theta^n_p \} \|_S \right) + \eta^n_3 \| \{ \theta^n_p \} \|^2_S ,
\]

with \( C_2 = \| \partial^2_{\Delta t} \theta^{n-1}_u \|^2_H + \| \{ \partial_{\Delta t} \theta^{n-1}_u \} \|^2_X + \| \eta^n_3 \|^2_X + \| \{ \theta^n_p \} \|^2_S \).

Estimating \( b(u,h, \theta^n_p) \) in (43a) gives

\[
\| \{ \theta^n_p \} \|_S \leq C_p \| \{ \eta^n_0 \} \|_H + \| \{ \theta^n_u \} \|_X ,
\]

and choosing \( q_h = \{ \theta^n_p \} \) in (43b) results in

\[
\| \{ \theta^n_p \} \|^2_X \leq \| \{ \theta^n_p \} \|^2_S + \| \{ \eta^n_0 \} \|^2_H + \| \{ \theta^n_u \} \|^2_X .
\]

From the last two inequalities

\[
\| \{ \theta^n_p \} \|^2_S \leq \| \partial^2_{\Delta t} \theta^n_u \|^2_H + \| \{ \partial_{\Delta t} \theta^n_u \} \|^2_X + \| \{ \theta^n_u \} \|^2_X + S^n + T^n .
\]

and combining (44), (45), (46) and choosing \( \eta_1, \eta_2, \eta_3 \in (0, 1) \) sufficiently small, we obtain

\[
\| \{ \theta^n_p \} \|^2_S \leq C_1 + C_2 + \| \eta^n_3 \|^2_X + \Delta t \sum_{k=3}^{n-1} (T^k + s^k) \\
+ \Delta t \sum_{k=3}^{n-1} \left( \| \{ \theta^n_p \} \|^2_Q + \| \partial^2_{\Delta t} \theta^n_u \|^2_H + \| \{ \partial_{\Delta t} \theta^n_u \} \|^2_X + \| \{ \theta^n_u \} \|^2_X \right) .
\]

Now, we estimate \( \eta^n_3 \) by Lemma 12 and Lemma 13 and by Sobolev’s embedding theorem [16, Th. 5.9.2]

\[
\eta^n_3 \leq (\Delta t)^4 \| u \|^2_{W^{4,\infty}(0,T;X)} + \| \theta_u \|^2_{W^{2,\infty}(0,T;X)} \leq (\Delta t)^4 \| u \|^2_{H^5(0,T;X)} + \| \theta_u \|^2_{H^3(0,T;X)} .
\]

Together with (39) and (41) follows

\[
\| \{ \theta^n_p \} \|^2_Q + \| \partial^2_{\Delta t} \theta^n_u \|^2_H + \| \{ \partial_{\Delta t} \theta^n_u \} \|^2_X + \| \{ \theta^n_u \} \|^2_X \\
\leq C_1 + C_2 + (\Delta t)^4 \| u \|^2_{H^5(0,T;X)} + \| \theta_u \|^2_{H^3(0,T;X)} \\
+ \Delta t \sum_{k=3}^{n-1} \left( \| \{ \theta^n_p \} \|^2_Q + \| \partial^2_{\Delta t} \theta^n_u \|^2_H + \| \{ \partial_{\Delta t} \theta^n_u \} \|^2_X + \| \{ \theta^n_u \} \|^2_X \right) .
\]

Finally, applying the discrete Gronwall’s lemma (Lemma 10) gives for \( t_n \leq T \)

\[
\| \{ \theta^n_p \} \|^2_Q + \| \partial^2_{\Delta t} \theta^n_u \|^2_H + \| \{ \partial_{\Delta t} \theta^n_u \} \|^2_X + \| \{ \theta^n_u \} \|^2_X \leq C_3 + (\Delta t)^4 \| u \|^2_{H^5(0,T;X)} + \| \theta_u \|^2_{H^3(0,T;X)} ,
\]

with a larger constant depending on \exp(T) and \( C_3 = C_1 + C_2 \). \qed
Together with Theorem 7 and Lemma 12, the final estimate follows.

**Corollary 15.** If \( u \in H^3(0, T; H^2(\Omega, \mathbb{R}^3)) \cap H^5(0, T; X) \) and \( p \in H^3(0, T; H^2(\Omega, \mathbb{R}) \cap Q) \) we have in case of sufficiently accurate initial values and sufficiently accurate first steps

\[
\|\{\tilde{u}(t_n) - u^n_0\}\|_H + \|\{\tilde{u}(t_n) - v^n_0\}\|_X + \|\{u(t_n) - u^n_0\}\|_X + \|\{p(t_n) - p^n_0\}\|_Q
\leq C(u, p) h + (\Delta t)^2
\]

with \( C(u, p) \) independent of \( \kappa, \Delta t \) and \( h \).

### 6.3. A discrete energy estimate.

Lemma 11 directly yields the time-discrete analog to (17) and (18). For this purpose, we define for \( n \geq 0 \)

\[
E^n_{\text{wave}, h}(u^n_h, v^n_h) = \frac{1}{2} m(v^n_h, v^n_h) + \frac{1}{2} a(u^n_h, v^n_h),
\]

\[
E^n_h(u^n_h, v^n_h, p^n_h) = E^n_{\text{wave}, h}(u^n_h, v^n_h) + \Delta t \sum_{i=1}^n c_n(\{p^n_i\}, \{p^n_{i-1}\}).
\]

Let \( u^n_h, v^n_h, a^n_h \in X_h \) and \( p^n_h \in Q_h \) be a solution of (35) with homogeneous right-hand side \( \ell \equiv 0 \) and initial values \( (u^n_h, v^n_h) \in X_h \times X_h \). Then, for \( n > 0 \), we have energy conservation in the form

\[
E^n_h(u^n_h, v^n_h, p^n_h) = E^0_{\text{wave}, h}(u^0_h, v^0_h).
\]

\[
\partial_{\Delta t} E^n_{\text{wave}, h}(u^n_h, v^n_h) = -c_n(\{p^n_n\}, \{p^n_{n-1}\}) \leq 0.
\]

In particular, in the undamped case \( \kappa = 0 \), energy would be conserved exactly.

A discrete interpretation of the Lagrange principle in Sect. 3.7 is not obvious.

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### References


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